Uniform Bounded Principle

Thm → Let B be a Banach Space and N a NLS. If $\{T_i\}$ is a non-empty set of bdd. (or Cont.) L.Ts. of B into N having the property that $\{T_i(x)\}$ is a bounded subset of N for each vector x in B, then $\{\|T_i\|\}$ is a bounded set of numbers i.e. $\{T_i\}$ is a subset of $\mathcal{B}(B, N)$.

Pf → Let us define →

$$F_n = \{x : x \in B \;\&\; \|T_i(x)\| \leq n, \forall i\} \quad \text{---(1)}$$

for each +ve integer 'n'.

Then we shall show that F_n is a closed subset of B.

$$x \in F_n \Leftrightarrow \|T_i(x)\| \leq n \quad \forall i$$

$$\Leftrightarrow T_i(x) \in S_n^c \quad \forall i$$

, where S_n^c denotes the closed sphere in N with centre '0' & radius 'n'

$$\Leftrightarrow x \in T_i^{-1}[S_n^c] \quad \forall i$$

$$\Leftrightarrow x \in \bigcap_i T_i^{-1}[S_n^c]$$

$$\therefore F_n = \underbrace{\bigcap_i T_i^{-1}[S_n^c]}$$

closed as intersection of closed sets is closed

$\therefore F_n$ is closed.

[Note: $\underbrace{S_n^c \text{ closed}}_{\text{in } N}$ & T_i cont. $\Rightarrow$ $\underbrace{T_i^{-1}[S_n^c]}_{\text{in } B}$ is also closed]

Further,

$$B = \bigcup_{n=1}^{\infty} F_n$$

If $B \neq \bigcup_{n=1}^{\infty}$ then $\exists$ some $x \in B$ s.t. $x \notin F_n$ for any n.

$\Rightarrow \|T_i(x)\| > n \quad \forall n$, for some i

$\Rightarrow$ the set $\{T_i(x)\}$ is not bdd.

which is a contradiction as $\{T_i\}$ is a non-empty set of bdd. transformations

$\therefore$ we must have —

$$B = \bigcup_{n=1}^{\infty} F_n$$

so that the complete space B is the union of a seqn of its subsets $F_n, \forall n$.

$\therefore$ By Baire-Category thm, $\exists$ an integer n_0 s.t. $\overline{F_{n_0}}$ has non-empty interior.

Since F_{n_0} is closed,

$$\therefore \overline{F_{n_0}} = F_{n_0}$$

i.e. closure of $\overline{F_{n_0}}$ is equal to F_{n_0}.

$\therefore$ F_{n_0} must have non-empty interior i.e. $\exists$ some $x_0 \in F_{n_0}$ s.t. F_{n_0} is a nbd. of x_0

Since F_{n_0} is closed, $\exists$ a closed sphere

$S = \{x \in B : \|x - x_0\| \leq r_0\} \subset F_{n_0}$ → (2)

[centre x_0 & radius r_0]

Now if $\|y\| \leq 1$, then for arbitrary but fixed i, we have →

$$\|T_i(y)\| = \left\|T_i\left(\frac{z}{r_0}\right)\right\| \; ; \quad \text{where } z = r_0 y$$

$$= \frac{1}{r_0}\|T_i(z)\|$$

$$= \frac{1}{r_0}\|T_i(z + x_0 - x_0)\|$$

$$= \frac{1}{r_0}\|T_i(z + x_0) - T_i(x_0)\|$$

$$\leq \frac{1}{r_0}\left[\|T_i(z + x_0)\| + \|-T_i(x_0)\|\right]$$

$$= \frac{1}{r_0}\left[\|T_i(z + x_0)\| + \|T_i(x_0)\|\right]$$

$$\leq \frac{1}{r_0}(n_0 + n_0) = \frac{2n_0}{r_0} \quad \to (3)$$

; $\because z + x_0 \in F_{n_0}$ & $x_0 \in F_{n_0}$

we have →

$$\|z + x_0 - x_0\| = \|z\| = \|r_0 y\| = r_0\|y\| \leq r_0$$

as $\|y\| \leq 1$

$\therefore z + x_0 \in S \subset F_{n_0}$.

& $x_0 \in S \subset F_{n_0}$

$\therefore \|T_i(y)\| \leq \dfrac{2n_0}{r_0}$ if $\|y\| \leq 1$

$\therefore \|T_i\| = \sup\{\|T_i(y)\| : \|y\| \leq 1\} \leq \dfrac{2n_0}{r_0}$

$\therefore \{\|T_i\|\}$ is a bdd. set of numbers. (Proved)